The $RO(C_2)$-graded cohomology of C_2-surfaces

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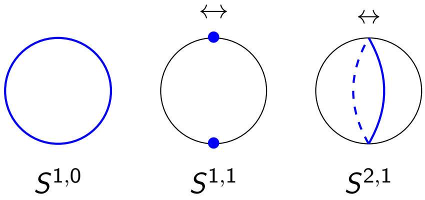
Introduction to $RO(G)$-graded Cohomology

- Let G be a finite group
- G-Top
- Given a finite-dimensional, real, orthogonal G-representation V, we can form the representation sphere $S^V = \hat{V}$
- Have equivariant suspensions $\Sigma^V X = S^V \wedge X$
- Bredon Cohomology $H^\alpha(-; M)$
  - $\alpha \in RO(G)$, M Mackey functor
  - Have suspension isomorphisms $\tilde{H}^\alpha(X; M) \cong \tilde{H}^{\alpha+V}(\Sigma^V X; M)$

The case when $G = C_2$

- $V \cong \mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn}$
  - $RO(C_2)$-graded cohomology is a bigraded theory
  - Refer to first grading as "topological dimension" and second grading as "weight"
- Write $H^{p,q}(X; M)$ for $H^{\mathbb{R}^{p,q}}(X; M)$
- Write $S^{p,q}$ for $S^{\mathbb{R}^{p,q}}$



- We will be considering the constant Mackey functor $\underline{\mathbb{Z}/2}$

The cohomology of orbits in $\underline{\mathbb{Z}/2}$-coefficients

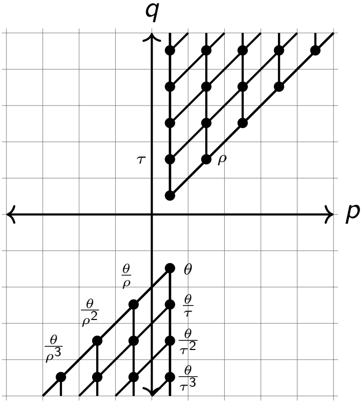
The (p, q) group is plotted in the box up and to the right of (p, q).

$\bullet = \mathbb{Z}/2$

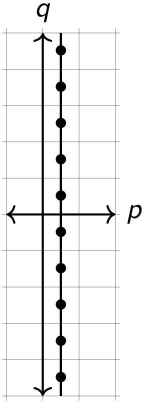
$| = \cdot\tau$

$/ = \cdot\rho$

$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{Z}/2})$



$A_0 = H^{*,*}(C_2; \underline{\mathbb{Z}/2})$



The cohomology of a point in $\underline{\mathbb{Z}/2}$-coefficients

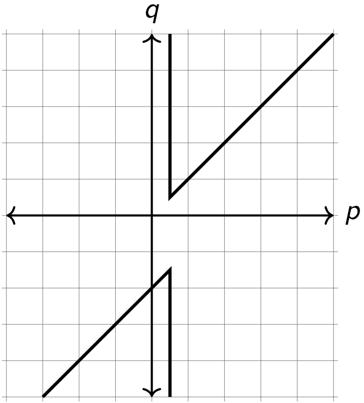
Abbreviated pictures

$\bullet = \mathbb{Z}/2$

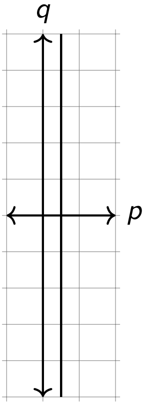
$| = \cdot\tau$

$/ = \cdot\rho$

$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{Z}/2})$



$A_0 = H^{*,*}(C_2; \underline{\mathbb{Z}/2})$

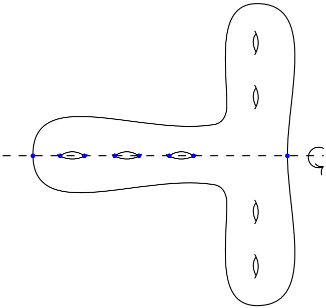


C_2-surfaces give a family of spaces to compute with

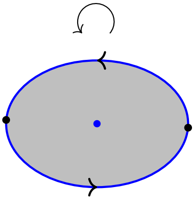
- In 2016, Dugger classified all C_2-surfaces up to equivariant isomorphism using equivariant surgery.

Examples:

$T_{7,2}^{spit}$



$\mathbb{R}P^2_{twist}$



The cohomology of non-free, non-trivial C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients

- Let X be a non-trivial, non-free C_2-surface
- $F = \#$ isolated fixed points, $C = \#$ fixed circles, $\beta = \dim_{\mathbb{Z}/2} H^1_{sing}(X; \mathbb{Z}/2)$

Theorem (H.)

There are two cases for the cohomology of X:

(i) *Suppose $C = 0$. Then*

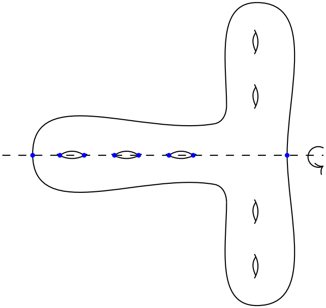
$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F-2} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta+2-F}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2$$

(ii) *Suppose $C \neq 0$. Then*

$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F+C-1} \oplus \left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C-1} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta+2-(F+2C)}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2$$

Return to our examples of C_2-surfaces

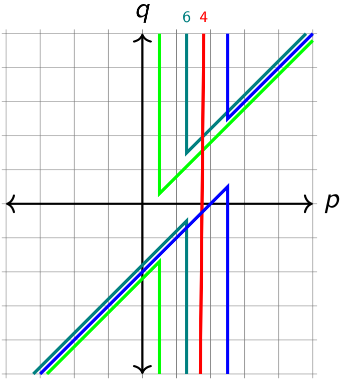
$T^{spit}_{7,2}$



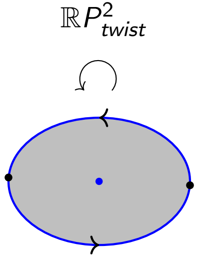
$F = 8, \quad C = 0, \quad \beta = 14$

$H^{*,*}(T^{spit}_{7,2}) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus 6}$

$\oplus \left(\Sigma^{1,0}A_0\right)^{\oplus 4} \oplus \Sigma^{2,2}\mathbb{M}_2$

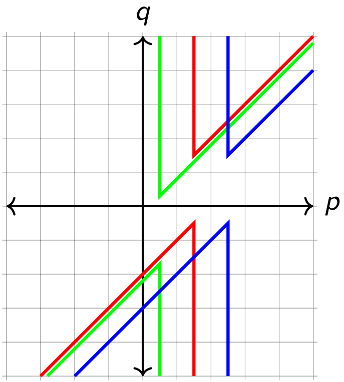


Return to our examples of C_2-surfaces



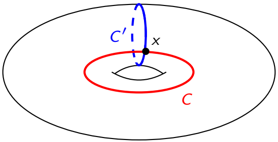
$$F = 1, \quad C = 1, \quad \beta = 1$$

$$H^{*,*}(\mathbb{R}P^2_{twist}) \cong \mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2$$



Nonequivariant fundamental classes

- M is a smooth n-dimensional manifold and N is a smooth k-dimensional submanifold
- Get a class $[N] \in H^{n-k}(M; \mathbb{Z}/2)$
- Example:



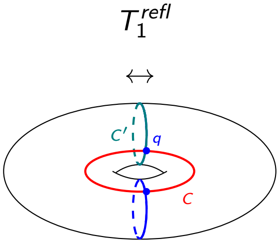
$$[C], [C'] \in H^1(T_1); [x] \in H^2(T_1)$$

$$[C] \smile [C'] = [C \cap C'] = [x]$$

$$H^*(T_1) \cong \mathbb{Z}/2[a, b]/(a^2 = b^2 = 0),$$

$$|a| = |b| = 1$$

An equivariant example

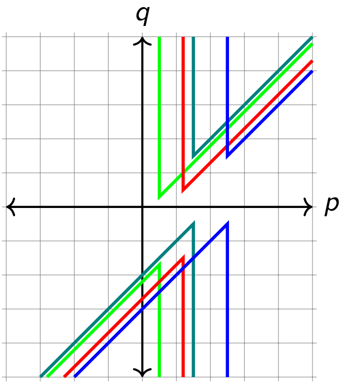


$F = 0, \quad C = 2, \quad \beta = 2$

$[C'] \in H^{1,1}, \quad [C] \in H^{1,0}$

$[q] \in H^{2,1}$

$[C'] \smile [C] = [C' \cap C] = [q]$



$H^{*,*}(T_1^{relf}) \cong$

$\mathbb{M}_2[a, b]/\langle a^2 = \rho \cdot a, b^2 = 0\rangle$

$|a| = (1, 1), \quad |b| = (1, 0)$

Equivariant fundamental classes

- X is a n-dimensional C_2-manifold and Y is a nonfree k-dimensional C_2-submanifold
- $[Y] \in H^{n-k,??}(X; \underline{\mathbb{Z}/2})$

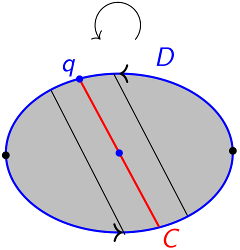
- Consider the equivariant normal bundle E of Y in X
- Over each fixed point $y \in Y^{C_2}$, $E_y \cong \mathbb{R}^{n-k,q_y}$
- Let q be the maximum weight appearing over Y^{C_2}

Theorem (H.)

We get a unique class $[Y] \in H^{n-k,q}(X; \underline{\mathbb{Z}/2})$

Example 2

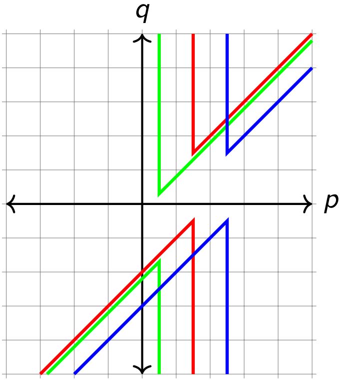
$\mathbb{R}P^2_{tw}$



$[C] \in H^{1,1} \quad [D] \in H^{1,1}$

$[q] \in H^{2,1}$

$[C] \smile [D] = \tau \cdot [q]$

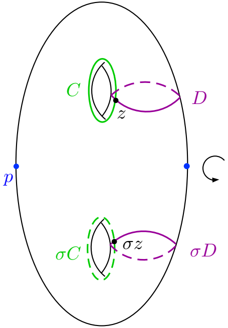


$\mathbb{M}_2[x, y]/(x^2 = \tau y + \rho x, xy = y^2 = 0),$

$|x| = (1, 1), |y| = (2, 1)$

Example 3

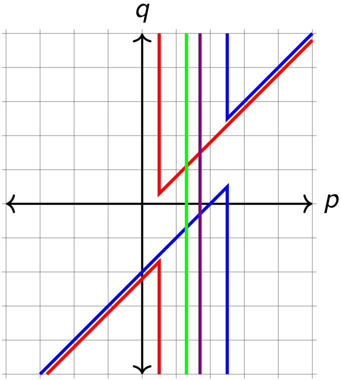
$T^{spit}_{2,1}$



$[C \sqcup \sigma C]_q, [D \sqcup \sigma D]_q \in H^{1,q}$

$[C \sqcup \sigma C]_r \smile [D \sqcup \sigma D]_s = [z \sqcup \sigma z]_{r+s}$

$F = 2, C = 0, \beta = 4$



Conclusions and next steps

- Have a working theory of fundamental classes with a nice intersection product

Theorem (H.)

The cohomology of all C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients is generated by fundamental classes.

- Can we say something similar for general C_2-manifolds?
- These classes were defined by proving there exists a "sort of Thom class" for C_2-vector bundles. Can we say more about this class?

Thank you!