Equivariant fundamental classes in $RO(C_2)$-graded cohomology

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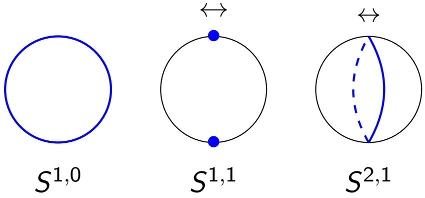
Introduction to $RO(G)$-graded Cohomology

- Let G be a finite group
- G-Top
- Given a finite-dimensional, real, orthogonal G-representation V, we can form the representation sphere $S^V = \hat{V}$
- Have equivariant suspensions $\Sigma^V X = S^V \wedge X$
- Bredon Cohomology $H^\alpha(-; M)$
  - $\alpha \in RO(G)$, M Mackey functor
  - Have suspension isomorphisms $\tilde{H}^\alpha(X; M) \cong \tilde{H}^{\alpha+V}(\Sigma^V X; M)$

The case when $G = C_2$

- $V \cong \mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn}$
  - $RO(C_2)$-graded cohomology is a bigraded theory
  - p is the "topological dimension", q is the "weight"
- $H^{p,q}(X; M) := H^{\mathbb{R}^{p,q}}(X; M)$
- $S^{p,q} := S^{\mathbb{R}^{p,q}}$



- Take $M = \underline{\mathbb{Z}/2}$

The cohomology of orbits in $\underline{\mathbb{Z}/2}$-coefficients

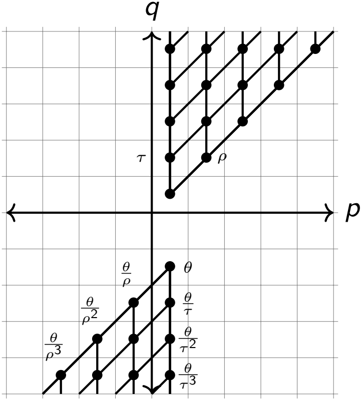
The (p, q) group is plotted in the box up and to the right of (p, q).

$\bullet = \mathbb{Z}/2$

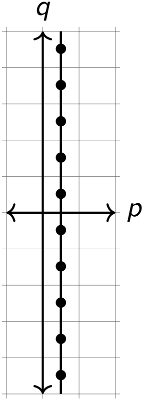
$| = \cdot\tau$

$/ = \cdot\rho$

$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{Z}/2})$



$A_0 = H^{*,*}(C_2; \underline{\mathbb{Z}/2})$



The cohomology of a point in $\underline{\mathbb{Z}/2}$-coefficients

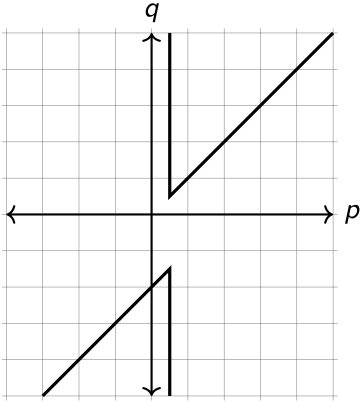
Abbreviated pictures

$\bullet = \mathbb{Z}/2$

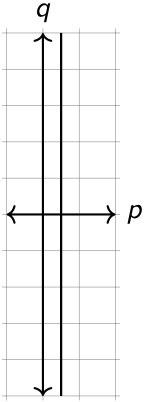
$\big| = \cdot\tau$

$/ = \cdot\rho$

$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{Z}/2})$



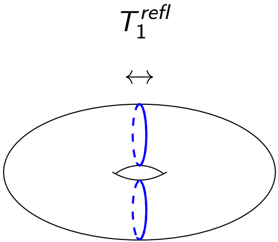
$A_0 = H^{*,*}(C_2; \underline{\mathbb{Z}/2})$

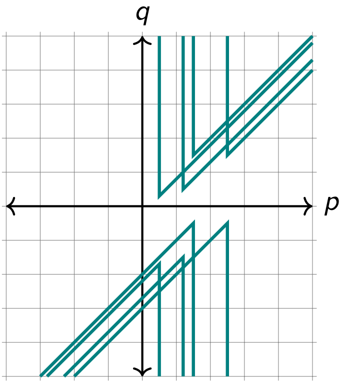


Prologue: C_2-surfaces

- Given a C_2-manifold, goal is to understand certain Bredon cohomology classes using equivariant submanifolds
- In 2016, Dugger classified all C_2-surfaces up to equivariant isomorphism using equivariant surgery.
- Computations have been done for all C_2-surfaces in $\underline{\mathbb{Z}/2}$ and $\underline{\mathbb{Z}}$ coefficients (H.)
- C_2-surfaces will be our examples in this talk

Example 1

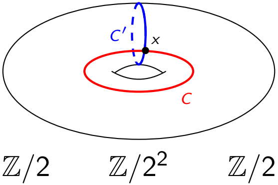




Q: How do we connect the algebraic answer to the geometry of the torus and its C_2-action?

Nonequivariant fundamental classes

- Example:



$$[C], [C'] \in H^1(T_1)$$

$$[C] \smile [C'] = [C \cap C'] = [x],$$

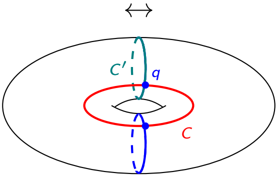
$$[C'] \smile [C'] = 0 = [C] \smile [C]$$

$$H^*(T_1) \cong \mathbb{Z}/2[a, b]/(a^2 = b^2 = 0),$$

- More generally, for $N^k \subset M^n$ smooth manifold, get fundamental class $[N] \in H^{n-k}(M)$
- Defined using the Thom isomorphism theorem for the normal bundle
- If X and Y intersect transversally, then $[X] \smile [Y] = [X \cap Y]$

Return to Example 1

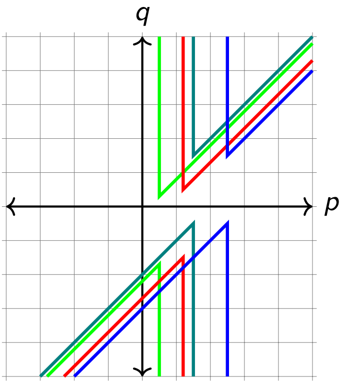
T_1^{refl}



$$[C'] \in H^{1,1}, \quad [C] \in H^{1,0}$$

$$[q] \in H^{2,1}$$

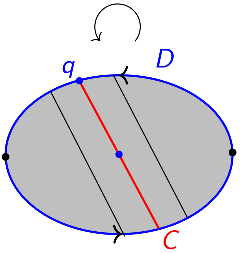
$$[C'] \smile [C] = [C' \cap C] = [q]$$

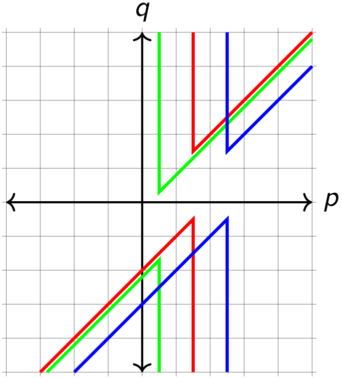


$$H^{*,*}(T_1^{relf}) \cong$$
$$\mathbb{M}_2[a, b]/\langle a^2 = \rho \cdot a, b^2 = 0\rangle$$
$$|a| = (1, 1), \quad |b| = (1, 0)$$

Example 2

$\mathbb{R}P^2_{tw}$





$[D] \in H^{1,1}$, $\quad [q] \in H^{2,1}$

$[C] \in H^{1,????}$

Equivariant fundamental classes

- X is a n-dimensional C_2-manifold and Y is a nonfree k-dimensional C_2-submanifold
- $[Y] \in H^{n-k,??}(X; \underline{\mathbb{Z}/2})$

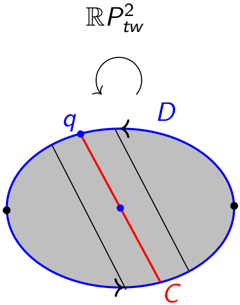
- Consider the equivariant normal bundle E of Y in X
- Over each fixed point $y \in Y^{C_2}$, $E_y \cong \mathbb{R}^{n-k,q_y}$
- Let q be the maximum weight appearing over Y^{C_2}

Theorem (H.)

We get a unique class $[Y] \in H^{n-k,q}(X; \underline{\mathbb{Z}/2})$

- Defined by proving an equivariant version of the Thom isomorphism theorem

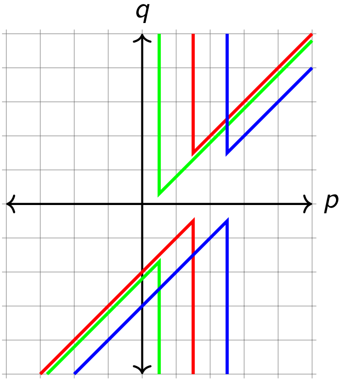
Return to Example 2



$$[D] \in H^{1,1}, \qquad [q] \in H^{2,1}$$

$$[C] \in H^{1,1}$$

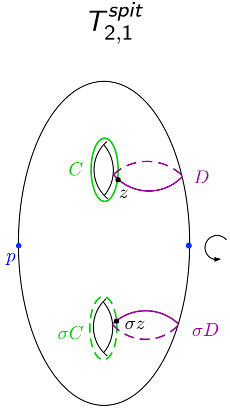
$$[C] \smile [D] = \tau \cdot [q]$$

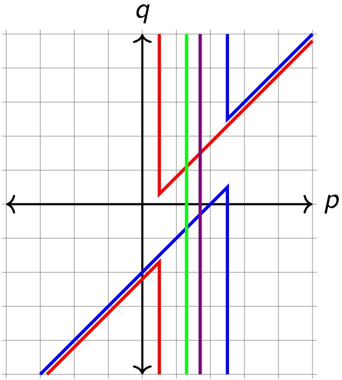


$$\mathbb{M}_2[x, y]/(x^2 = \tau y + \rho x, xy = y^2 = 0),$$

$$|x| = (1, 1), |y| = (2, 1)$$

Example 3





$$[C \sqcup \sigma C]_q, [D \sqcup \sigma D]_q \in H^{1,q}$$

$$[C \sqcup \sigma C]_r \smile [D \sqcup \sigma D]_s = [z \sqcup \sigma z]_{r+s}$$

Conclusions

- Have a working theory of fundamental classes with a nice intersection product

Theorem (H.)

The cohomology of all C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients is generated by fundamental classes.

- Future question(s): equivariant analog of Steenrod's problem:
- Q: Is every class in $H_{*,*}(X;\underline{\mathbb{Z}/2})$ an $\mathbb{M}_2$-multiple of $f_*[M]$ for some manifold M and $f : M \to X$?

Thank you!