# Research Statement

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## 1. Introduction

My research is in equivariant algebraic topology. Loosely speaking, algebraic topology studies ways to assign meaningful algebraic data to geometric objects. Then we can use algebraic tools to answer geometric questions that were otherwise inaccessible. This interplay between algebra and geometry has led to rich mathematical results, and the tools developed in algebraic topology continue to be useful in many areas of mathematics.

*Equivariant* algebraic topology studies ways to assign algebraic data to geometric objects that have a chosen symmetry. When this assignment satisfies certain nice properties, we call it an *equivariant cohomology theory.* Equivariant cohomology theories were introduced over three decades ago [1, 24], and gained a recent resurgence of attention inspired by their use in the breakthrough solution to the Kervaire invariant one problem [22]. Equivariant tools are also used in algebraic $K$-theory computations [10, 20, 21], connecting them to algebraic geometry and number theory. This an active and exciting area of mathematics, and equivariant theories are now vital tools in algebraic topology.

In classical algebraic topology one of the most widely used algebraic invariants is singular cohomology. This theory has been studied extensively over the past century. We know how to compute this algebraic data for a given object, and more so, we understand how certain geometric properties are detected by the answer. The equivariant analog of singular cohomology is *Bredon cohomology.* However the complexity of its algebraic outputs make computations difficult, and for many groups of symmetries, even the Bredon cohomology of a point remains unknown.

Due to its relative complexity, many familiar results from singular cohomology have yet to be replicated in Bredon cohomology. Much of my work focuses on Bredon cohomology for spaces with order two symmetries, that is, spaces with an action of $C_2$, the cyclic group of order two. Subtleties already arise in this first nontrivial case, and $C_2$-equivariant cohomology theories have useful connections to other areas of mathematics, such as algebraic geometry over $\mathbb{R}$ (see [23, 19, 4]).

Many of my projects are motivated by two goals. One is to compute the cohomology of different families of $C_2$-spaces to increase our encyclopedia of examples. The second is to develop a geometric understanding of these answers that builds our intuition and can be applied beyond the currently known computations. These goals led to my published papers [18, 17] and my preprint [16]. See Section 2 for a summary of my past and ongoing computational projects.

In addition to Bredon cohomology computations, I have a few collaborative projects that use *algebraic models* to translate information about a given equivariant cohomology theory into the purely algebraic world of chain complexes. In these projects we use homological algebra techniques to prove properties about the corresponding cohomology theory. In joint work with Dugger and C. May we use this method to prove a useful classification result for Bredon cohomology that helps predict the types of outputs one can get from this theory. These results are in our preprint [14]. My joint work with Bohmann, Ishak, Kędziorek, and May used algebraic models to establish a uniqueness result about another important equivariant cohomology theory, equivariant $K$-theory, where the group of symmetries was any finite abelian group. These results are published in two papers [8, 9]. Section 3 gives an overview of these projects and their future directions.

## 2. Bredon cohomology computations and interpretations

Given a topological space $X$ and an abelian group $A$, we can consider the singular cohomology of $X$ in $A$-coefficients. This will be a sequence of abelian groups denoted by $H^n(X;A)$ for $n \in \mathbb{Z}$, and the choice of coefficients tells us the cohomology of a single point. Bredon cohomology is the equivariant analog of singular cohomology, and when the group of equivariance is the trivial group, Bredon cohomology coincides with singular cohomology. However both the grading and the coefficients are more complicated for nontrivial groups.

Let $G$ be a finite group. Recall any discrete $G$-set can be written as a disjoint union of orbits, each of which is isomorphic, as a $G$-set, to a set of cosets $G/H$ for some subgroup $H \leq G$. In other words, we have different types of "points" coming from the different types of orbits. The coefficients $\underline{M}$ thus consist of a family of abelian groups $\underline{M}(G/H)$, one for each orbit $G/H$, as well as various maps between them. We call this data a *Mackey functor*. This should be thought of as the equivariant analog of an abelian group, and it exactly encodes the 0th Bredon cohomology of the $G$-equivariant "points". For a Mackey functor $\underline{M}$ and a $G$-space $X$, the Bredon cohomology is then a family of abelian groups indexed by the real representation ring for $G$.

For $C_2$, the cyclic group of order two, any such representation decomposes as a direct sum of trivial and sign representations. Thus the Bredon cohomology can be regarded as a bigraded abelian group. We write $H^{p,q}(X;\underline{M})$ for the cohomology of a $C_2$-space $X$ with $\underline{M}$-coefficients. The value $p$ is the dimension of the representation and called the *topological degree*, and the value $q$ is the number of sign representations and called the *weight*. You can think of this as a bigraded version of singular cohomology, where the second grading is needed to keep track of the $C_2$-action.

2.1. **Surface computations.** In my thesis I computed the Bredon cohomology of all $C_2$-surfaces. A *$C_2$-surface* is a closed 2-manifold with a chosen order two symmetry, that is, a surface with a continuous action by the cyclic group of order two. For example, the sphere has mirror symmetry given by reflecting two hemispheres, and the torus has rotational symmetry given by rotating 180°. Figure 1 shows a few other examples of $C_2$-surfaces where the fixed set is drawn in blue.

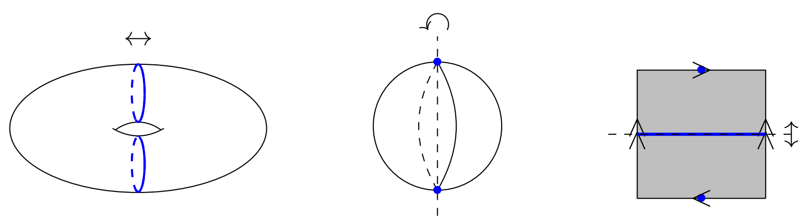

Figure 1. Examples of $C_2$-surfaces.

Dugger classified all $C_2$-surfaces up to equivariant homeomorphism in 2016 [13]. The family of $C_2$-surfaces is simple enough to keep computations manageable, yet rich enough to shed light on the geometric properties detected by Bredon cohomology. I computed their cohomology with coefficients given by the constant Mackey functors $\underline{\mathbb{Z}/2}$ and $\underline{\mathbb{Z}}$, and furthermore found an algorithmic way to present the answer that depends on only a few invariants of the surface and its action.

**Result 2.2.** [18, 16] *I produced an algorithm that will give the Bredon cohomology of any $C_2$-surface with $\underline{\mathbb{Z}/2}$-coefficients and with $\underline{\mathbb{Z}}$-coefficients. There are sixteen different cases. In each case, the cohomology depends only on a few properties of the $C_2$-space, and the answer is entirely formulaic.*

The rigorous statement for $\underline{\mathbb{Z}/2}$ is published in [18] and the statement for $\underline{\mathbb{Z}}$ is in the preprint [16]. My computations led me to conjecture and prove the following fact about the cohomology of $C_2$-manifolds of any dimension, not just surfaces. This is the equivariant version of the Poincaré duality statement, *"If $X$ is a closed $n$-manifold, then $H^n(X;\mathbb{Z}/2) = \mathbb{Z}/2$ while $H^j(X;\mathbb{Z}/2) = 0$ for $j > n$."* Below $\mathbb{M}_2$ denotes the bigraded cohomology of a point with $\underline{\mathbb{Z}/2}$-coefficients.

**Theorem 2.3.** [18, Thm A.1] *Let $X$ be an $n$-dimensional, closed $C_2$-manifold with a nonfree action. Suppose $n-k$ is the largest dimension of submanifold appearing as a component of the fixed set. There is exactly one summand of $H^{*,*}(X;\underline{\mathbb{Z}/2})$ of the form $\Sigma^{i,j}\mathbb{M}_2$ where $i \geq n$, and it occurs for $(i,j) = (n,k)$.*

The computations with $\underline{\mathbb{Z}}$-coefficients suggest a similar statement for $C_2$-manifolds whose underlying space is orientable. This is stated below. Here $\mathbb{M}$ denotes the bigraded cohomology of a point with $\underline{\mathbb{Z}}$-coefficients.

**Conjecture 2.4.** *Let $X$ be an $n$-dimensional, closed $C_2$-manifold with a nonfree action whose underlying space is an orientable manifold. Suppose $n-k$ is the largest dimension of submanifold appearing as a component of the fixed set. There is exactly one summand of $H^{*,*}(X;\underline{\mathbb{Z}})$ of the form $\Sigma^{i,j}\mathbb{M}$ where $i \geq n$, and it occurs for $(i,j)=(n,k)$.*

The rings $\mathbb{M}_2 = H^{*,*}(pt;\underline{\mathbb{Z}/2})$ and $\mathbb{M} = H^{*,*}(pt;\underline{\mathbb{Z}})$ are bigraded, commutative, and non-Noetherian. Thus modules over these rings can be complex and mysterious. C. May showed in [25] that $\mathbb{M}_2$ is injective as a module over itself, and I used this property in the proof of Theorem 2.3. However this property does not hold for $\mathbb{M}$, so we cannot apply the same techniques to solve Conjecture 2.4. Even though $\mathbb{M}$ is not self-injective, in many computations it behaves as if it were. The algebraic properties of $\mathbb{M}$ require further investigation, and I plan to study these properties in future projects.

2.5. **Fundamental classes for submanifolds.** Computing algebraic data is only one step in algebraic topology. We next want to determine which geometric properties are detected with this algebraic information. One geometric interpretation of singular cohomology involves special elements of the cohomology data called *fundamental classes*. Fundamental classes exist for manifolds, and these classes give us information about submanifolds and how they intersect.

Using my computations for $C_2$-surfaces, I was able to construct an analog of fundamental classes in Bredon cohomology. One way to define the singular cohomology fundamental classes uses a classical theorem known as the Thom isomorphism theorem. There are counterexamples that show no direct analog of the Thom isomorphism theorem can exist for Bredon cohomology, at least not for general $C_2$-vector bundles with $\underline{\mathbb{Z}/2}$-coefficients. Though, based on my computations for surfaces, I was able to conjecture and prove a weaker version of this theorem. Below is a summary of what I proved.

**Result 2.6.** [17] *There is a weak version of the Thom isomorphism theorem in Bredon cohomology with $\underline{\mathbb{Z}/2}$-coefficients that allows one to define fundamental classes for $C_2$-submanifolds of $C_2$-manifolds. These fundamental classes share many properties with the singular cohomology fundamental classes, including having an intersection product.*

These equivariant fundamental classes give a way to geometrically interpret the $C_2$-surface computations. In future work, I plan to explore how much of this holds for coefficient systems other than $\underline{\mathbb{Z}/2}$. I also plan to explore how much of this works for other groups of equivariance, starting with cyclic groups of odd prime order.

**Problem 2.7.** *Prove a version of the Thom isomorphism theorem for $C_2$-vector bundles with coefficients given by more general Mackey functors.*

**Problem 2.8.** *Formulate and prove a version of the Thom isomorphism theorem for $C_p$-vector bundles in Bredon cohomology with $\underline{\mathbb{Z}/p}$ coefficients.*

2.9. **Configuration spaces.** My current computational project investigates the Bredon cohomology of *configuration spaces*. Given a topological space $X$ we define the *ordered configuration space* of $n$ points in $X$ to be

$$\mathrm{Conf}_n(X) := \{(x_1,\ldots,x_n) \in X^n \colon x_i \neq x_j \text{ for } i \neq j\}.$$

Note this space has an action of the symmetric group $\Sigma_n$ given by permuting the points. The *unordered configuration space* is then defined to be the orbit space $\mathrm{Conf}_n(X)/\Sigma_n$. Configuration spaces appear in many areas of mathematics, and their singular cohomology when $X$ is a manifold has been investigated in many papers, see [2, 6, 7, 11, 12] for just some examples.

If $X$ is a $C_2$-space, then the configuration space inherits a $C_2$-action. In joint work with Dugger, we are computing the Bredon cohomology of families of such configuration spaces, starting with configurations of points in $C_2$-representations.

**Problem 2.10.** *Let $V$ be a finite-dimensional, real $C_2$-representation. Compute the Bredon cohomology of the ordered and unordered configuration spaces of points in $V$.*

In future work we plan to investigate the cohomology of configurations of points on $C_2$-surfaces.

**Problem 2.11.** *Let $X$ be a closed surface with a $C_2$-action. Describe the Bredon cohomology of the configuration space of points in $X$.*

## 3. Algebraic models

One approach to understanding a cohomology theory is to do computations and then look for patterns. Another approach is to study the cohomology theory itself and how it interacts with other theories. This is done using the *equivariant stable homotopy category.* Each cohomology theory corresponds to an *equivariant spectrum* in this category. An advantage of working in this setting is that we can use algebraic models for certain families of spectra.

3.1. **Classifying modules over Eilenberg–MacLane Spectra.** For a Mackey ring $\underline{R}$, Bredon cohomology with $\underline{R}$-coefficients is represented by the equivariant Eilenberg–MacLane sprectrum $H\underline{R}$. Schwede and Shipley showed that the category of $H\underline{R}$-modules is homotopically equivalent to the category of chain complexes of $\underline{R}$-modules [26]. Loosely speaking, this means we can study the cohomology theory given by $H\underline{R}$ (so Bredon cohomology with $\underline{R}$-coefficients) by instead working in the purely algebraic world of chain complexes of $\underline{R}$-modules.

C. May showed in [25] that the Bredon cohomology with $\underline{\mathbb{Z}/2}$-coefficients of any finite $C_2$-CW complex is isomorphic to a direct sum of free modules and one other type of module. This result was surprising because the cohomology of a point is a non-Noetherian, bigraded ring, and thus modules over this ring can be quite complicated. The strength of May's result suggested a decomposition is occurring on the spectrum level. Dugger, May, and I investigated this spectrum level question by studying the category of chain complexes of $\underline{\mathbb{Z}/2}$-modules. We proved the following.

**Theorem 3.2.** [14] *Every bounded chain complex of $\underline{\mathbb{Z}/2}$-modules is quasi-isomorphic to a direct sum of chain complexes such that each summand is in one of four families of simple chain complexes.*

From this, we were able to conclude the following result in equivariant homotopy theory.

**Theorem 3.3.** [14] *Every compact $H\underline{\mathbb{Z}/2}$-module is weakly equivalent to a wedge sum of suspensions of modules of the form $H\underline{\mathbb{Z}/2}$, $H\underline{\mathbb{Z}/2} \wedge \Sigma^\infty S^n_{a+}$ where $S^n_a$ is the $n$-sphere with the antipodal action, and the cofiber of elements $\tau^i$ for a fixed element $\tau \in H^{0,1}(pt; \underline{\mathbb{Z}/2})$.*

In addition, we give a concrete description of the derived category $\mathcal{D}(\underline{\mathbb{Z}/2})$, including computing its Balmer spectrum as introduced in [3]. In future work, we plan to consider the following problem.

**Problem 3.4.** *For $G = C_p$, understand the derived category of bounded chain complexes of $\underline{\mathbb{Z}/p}$-modules and use this to give a description of compact $H\underline{\mathbb{Z}/p}$-modules up to weak equivalence.*

If we can understand this category and prove a result similar to Theorem 3.2, then we could get a structure theorem for $C_p$-equivariant Bredon cohomology that would greatly simplify computations.

3.5. **Rational equivariant $K$-theory.** A key ingredient in the solution to the Kervaire invariant one problem was the existence of "norm maps" [22]. These norm maps encode different levels commutativity for ring objects in the equivariant stable homotopy category. The rough idea is that ring objects equipped with these norm maps have a commutative multiplication that is highly compatible with the group action. When a ring spectrum has all possible norm maps we call it a *genuine commutative ring spectrum.* The goal of this project is to understand the structure of these norm maps in important examples of equivariant cohomology theories.

Fix a finite group $G$. As a starting point, one can try to understand norm maps rationally using algebraic models. Nonequivariantly, rational spectra are homotopically equivalent to rational chain complexes. Greenlees and May gave an algebraic model for equivariant rational spectra where the objects are products of rational chain complexes [15]. The product is indexed by conjugacy classes of subgroups $H \leq G$, and each factor also has an action by the Weyl group $W_G H = N_G H/H$ where $N_G H$ denotes the normalizer of $H$ in $G$. That is, we can study rational $G$-spectra using the model

$$\prod_{(H)\leq G} \mathrm{Ch}(\mathbb{Q}[W_G H]\text{-mod}),$$

where $(H)$ denotes the conjugacy class for $H$.

Building on this, Wimmer gave a model for rational genuine $G$-commutative ring spectra [27]. Wimmer's model includes an algebra structure on each chain complex, together with maps between subconjugate entries that respect the multiplication and the differentials. The maps between subconjugate entries exactly encode the norm map structure.

In a joint project with Bohmann, Ishak, Kędziorek, and C. May, we use this algebraic model to study the norm maps in equivariant $K$-theory. Equivariant $K$-theory is an important cohomology theory that arises from considering isomorphism classes of equivariant vector bundles over a $G$-space. We first computed the image of this in the algebraic model. After computing this image, we noticed the corresponding family of chain complexes had some nice uniqueness properties. In particular, the norm maps on the chain complexes were determined by the ring structures on the resulting homologies. This translated to the following uniqueness result in the equivariant stable homotopy category.

**Result 3.6.** [9, 8] *The norm maps on rational equivariant $K$-theory are entirely forced by its (naive) commutative multiplication. That is, if $X$ is a genuine commutative ring spectrum that has the same naive multiplication on its homotopy groups as that of rational $K$-theory (so forgetting the data coming from the norm maps), then $X$ is weakly equivalent to rational $K$-theory as a genuine commutative ring spectrum.*

We also showed such a uniqueness statement fails for other genuine commutative ring spectra. For example, we showed this fails for the connective cover of rational equivariant $K$-theory.

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