The $RO(C_2)$-graded cohomology of C_2-surfaces

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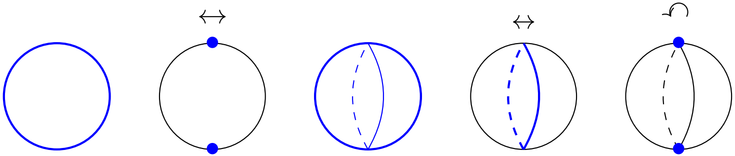
Introduction to $RO(G)$-graded Cohomology

- Let G be a finite group
- G-Top
- Given a finite-dimensional, real, orthogonal G-representation V, we can form the representation sphere $S^V = \hat{V}$
- Have equivariant suspensions $\Sigma^V X = S^V \wedge X$
- Bredon Cohomology $H^\alpha(-; M)$
  - $\alpha \in RO(G)$, M Mackey functor
  - Have suspension isomorphisms $\tilde{H}^\alpha(X; M) \cong \tilde{H}^{\alpha+V}(\Sigma^V X; M)$

The case when $G = C_2$

- Given a real representation V, $V \cong \mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn}$
  - $RO(C_2)$-graded cohomology is a bigraded theory
- We will write $H^{p,q}(X; M)$ for $H^{\mathbb{R}^{p,q}}(X; M)$
- For representation spheres, we will write $S^{p,q}$ for $S^{\mathbb{R}^{p,q}}$



$S^{1,0}$ $S^{1,1}$ $S^{2,0}$ $S^{2,1}$ $S^{2,2}$

- We will be considering two coefficient systems: the constant Mackey functors $\underline{\mathbb{Z}/2}$ and $\underline{\mathbb{Z}}$

The cohomology of a point in $\underline{\mathbb{Z}/2}$-coefficients

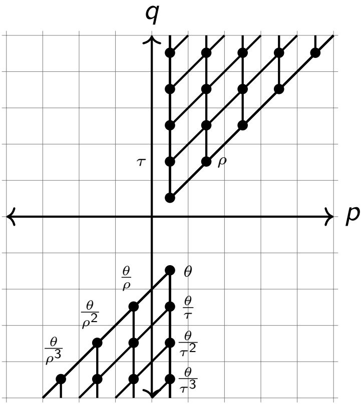
The (p, q) group is plotted in the box up and to the right of (p, q).

$\bullet = \mathbb{Z}/2$

$| = \cdot\tau$

$/ = \cdot\rho$

$H^{*,*}(pt; \underline{\mathbb{Z}/2})$

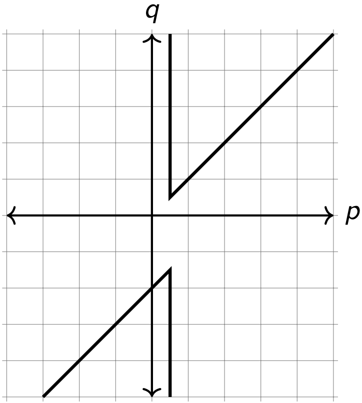


- Write $\mathbb{M}_2$ for $H^{*,*}(pt; \underline{\mathbb{Z}/2})$
- We have a map $X \to pt$ which makes $H^{*,*}(X; \underline{\mathbb{Z}/2})$ an $\mathbb{M}_2$-module
- We are interested in $H^{*,*}(X; \underline{\mathbb{Z}/2})$ as an $\mathbb{M}_2$-module

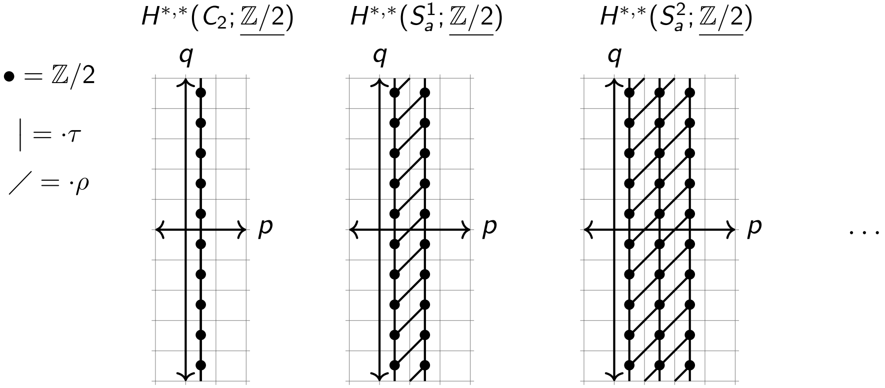
The cohomology of a point in $\underline{\mathbb{Z}/2}$-coefficients

Abbreviated drawing for $\mathbb{M}_2$

$H^{*,*}(pt; \underline{\mathbb{Z}/2})$



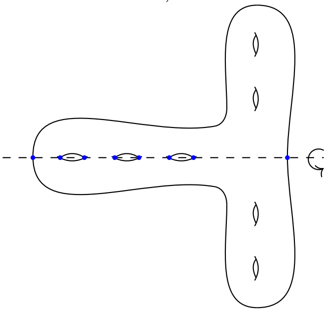
The free orbit and the antipodal spheres



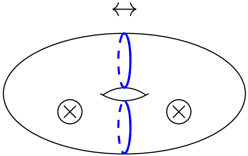
- Write A_n for $H^{*,*}(S^n_a; \underline{\mathbb{Z}/2})$

Some examples of C_2-surfaces

$T^{spit}_{7,2}$



$T^{refl}_1 \#_2 \mathbb{R}P^2$



- In 2016, Dugger classified all C_2-surfaces up to equivariant isomorphism using equivariant surgery.

The cohomology of non-free, non-trivial C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients

- Let X be a non-trivial, non-free C_2-surface
- $F = \#$ isolated fixed points, $C = \#$ fixed circles, $\beta = \dim_{\mathbb{Z}/2} H^1_{sing}(X; \mathbb{Z}/2)$

Theorem (H.)

There are two cases for the cohomology of X:

(i) *Suppose $C = 0$. Then*

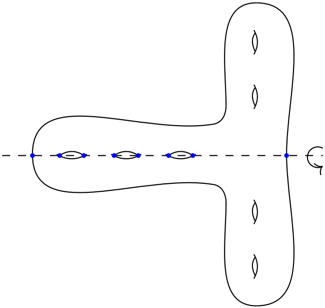
$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F-2} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta+2-F}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2$$

(ii) *Suppose $C \neq 0$. Then*

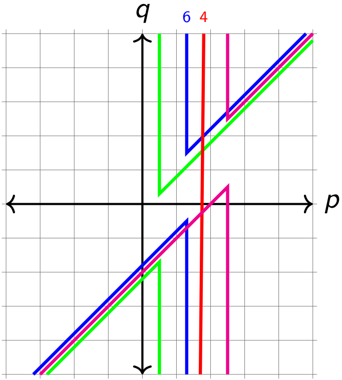
$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F+C-1} \oplus \left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C-1} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta+2-(F+2C)}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2$$

Return to our examples of C_2-surfaces

$T^{spit}_{7,2}$



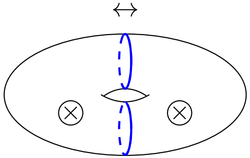
$F = 8, C = 0, \beta = 14$



$H^{*,*}(T^{spit}_{7,2}) \cong \mathbb{M}_2 \oplus (\Sigma^{1,1}\mathbb{M}_2)^{\oplus 6}$
$\oplus (\Sigma^{1,0}A_0)^{\oplus 4} \oplus \Sigma^{2,2}\mathbb{M}_2$

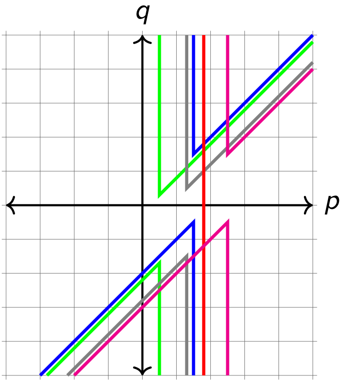
Return to our examples of C_2-surfaces

$T_1^{refl} \#_2 \mathbb{R}P^2$

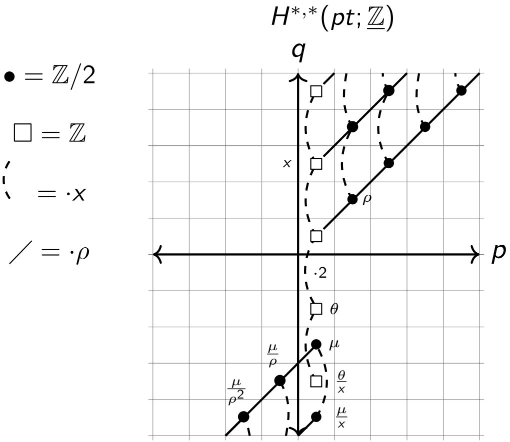


$F = 0, C = 2, \beta = 4$

$H^{*,*}(T_1^{refl} \#_2 \mathbb{R}P^2) \cong \mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2$
$\oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{1,0}A_0 \oplus \Sigma^{2,2}\mathbb{M}_2$

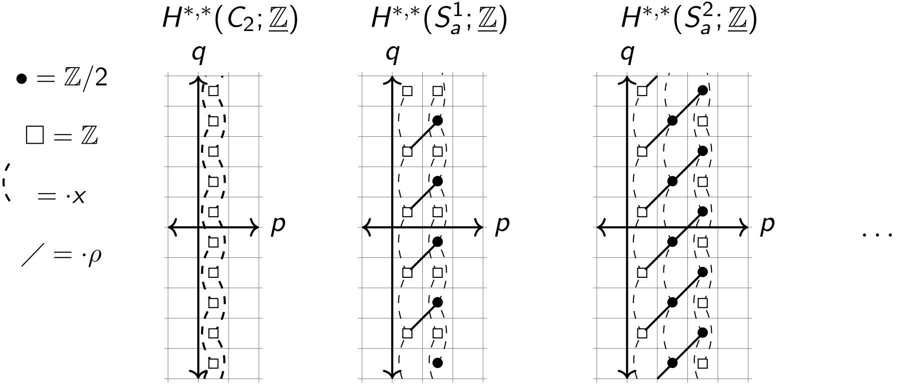


The cohomology of a point in $\underline{\mathbb{Z}}$-coefficients



- Write $\mathbb{M}$ for the ring $H^{*,*}(pt; \underline{\mathbb{Z}})$
- Want to compute $H^{*,*}(X; \underline{\mathbb{Z}})$ as an $\mathbb{M}$-module

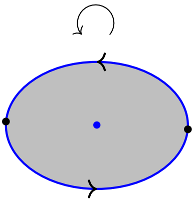
The cohomology of the free orbit and the antipodal spheres



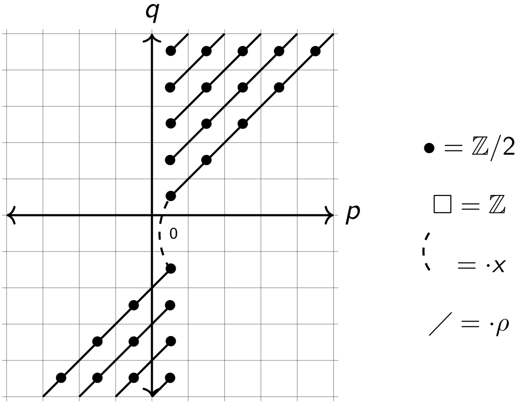
- Write $\mathbb{A}_n$ for the $\mathbb{M}$-module $H^{*,*}(S^n_a;\underline{\mathbb{Z}})$.

More $\mathbb{M}$-modules

$\mathbb{R}P^2_{twist}$



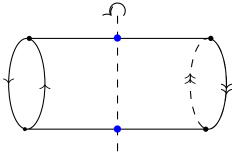
$\mathbb{B}_2$ (x-connections suppressed)



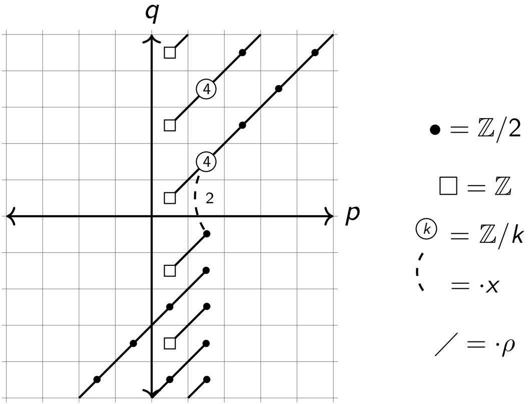
$$\tilde{H}^{*,*}(\mathbb{R}P^2_{twist}; \underline{\mathbb{Z}}) \cong \Sigma^{2,1}\mathbb{B}_2$$

More $\mathbb{M}$-modules

$S^{2,2}\#_2\mathbb{R}P^2$



$\mathbb{D}_4$ (x-connections suppressed)



$$\tilde{H}^{*,*}(S^{2,2}\#_2\mathbb{R}P^2) \cong \Sigma^{1,1}\mathbb{D}_4$$

The cohomology of non-free, non-trivial C_2-surfaces in $\underline{\mathbb{Z}}$-coefficients

- For orientable surfaces, the answer is the "same" as in $\underline{\mathbb{Z}/2}$-coefficients; just swap $\mathbb{M}_2$ for $\mathbb{M}$ and A_n for $\mathbb{A}_n$.
- Let X be a non-orientable surface with a non-free, non-trivial C_2-action
  - $F = \#$ isolated fixed points
  - $C = \#$ fixed circles
    - $C_+ = \#$ two-sided fixed circles
    - $C_- = \#$ one-sided fixed circles
  - $\beta = \dim_{\mathbb{Z}/2} H^1_{sing}(X; \underline{\mathbb{Z}/2})$

The cohomology of non-free, non-trivial, non-orientable C_2-surfaces in $\underline{\mathbb{Z}}$-coefficients

Theorem (H.)

There are three cases for the cohomology of X:

(i) *($F \neq 0$, $C = 0$)*

$$H^{*,*}(X) \cong \mathbb{M} \oplus \left(\Sigma^{1,0}\mathbb{A}_0\right)^{\oplus \frac{\beta - F}{2}} \oplus \left(\Sigma^{1,1}\mathbb{M}\right)^{\oplus F-2} \oplus \Sigma^{1,1}\mathbb{D}_4$$

(ii) *($F = 0$, $C_+ \neq 0$, $C_- = 0$)*

$$H^{*,*}(X) \cong \mathbb{M} \oplus \left(\Sigma^{1,0}\mathbb{A}_0\right)^{\oplus \frac{\beta - 2C_+}{2}} \oplus \left(\Sigma^{1,0}\mathbb{M}\right)^{\oplus C_+ - 1} \oplus \left(\Sigma^{1,1}\mathbb{M}\right)^{\oplus C_+ - 1} \oplus \Sigma^{1,0}\mathbb{D}_4$$

The cohomolgy of non-free, non-trivial, non-orientable C_2-surfaces in $\underline{\mathbb{Z}}$-coefficients

Theorem (H.)

There are three cases for the cohomology of X:

(iii) *($F \neq 0$, $C_+ \neq 0$, $C_- = 0$) or ($F \geq 0$, $C_+ \geq 0$, $C_- \neq 0$)*

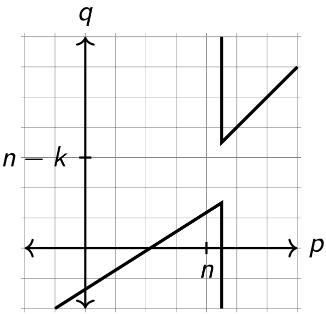
$$H^{*,*}(X) \cong \mathbb{M} \oplus \left(\Sigma^{1,0}\mathbb{A}_0\right)^{\oplus \frac{\beta+2-(F+2C)}{2}} \oplus \left(\Sigma^{1,0}\mathbb{M}\right)^{\oplus C-1} \oplus \left(\Sigma^{1,1}\mathbb{M}\right)^{\oplus F+C-2} \oplus \Sigma^{2,1}\mathbb{B}_2$$

Next steps: General statements about manifolds with C_2-actions

Theorem (Hogle, H.)

Any non-free C_2-manifold X has a top dimensional $\mathbb{M}_2$ generated at $(n, n-k)$

- $n = \dim(X)$
- $k =$ *max dimension of the fixed set*



- Similar statement for $\underline{\mathbb{Z}}$ if X is orientable.
- For non-orientable manifolds, there may be a similar statement with $\mathbb{B}_2$ or $\mathbb{D}_4$ instead of $\mathbb{M}$.

Next steps: Structure Theorem for $\underline{\mathbb{Z}}$-coefficients?

Theorem (C. May 2017)

For any finite C_2-CW complex X, we can decompose the $RO(C_2)$-graded cohomology of X with constant $\underline{\mathbb{Z}/2}$-coefficients as

$$H^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\oplus_i \Sigma^{p_i,q_i}\mathbb{M}_2) \oplus \left(\oplus_j \Sigma^{p_j,0} A_{n_j}\right)$$

as a module over $\mathbb{M}_2 = H^{*,*}(pt;\underline{\mathbb{Z}/2})$.

- Possible building blocks for a $\underline{\mathbb{Z}}$ structure theorem: $\mathbb{M}$, $\mathbb{A}_n$, $\mathbb{B}_{2k}$, $\mathbb{D}_{4k}$, coker $\left(\mathbb{M} \xrightarrow{2k+1} \mathbb{M}\right)$, a couple other pieces that appear for free actions.

Thank you!