Math 444: Final Project Information **Spring 2025**

Contents

Important Dates at a Glance

- Sunday, April 13... Project Preference Form due
- Sunday, April 20... Project Plan Form due
- Friday, May 2... Presentation and Paper Outlines due
- Sunday, May 4 or Tuesday, May 6... Practice Presentation Reflection due
- Tuesday, May 6 or Thursday, May 8... Presentation in class
- Friday, May 9... Written Project Draft due
- Wednesday, May 14... Final Written Project due

General Information

Your final project will consist of an expository paper on an algebraic topology topic as well as a 15-minute presentation introducing the topic to the class. This is an opportunity for you to improve your mathematical writing skills, independent study skills, and presentation skills. Your final project is worth 25% of your final course grade.

A good written final project will:

- cover an algebraic topology topic that uses a significant amount of our course material as background (in particular, you must use one of the algebraic invariants we learned in the second half of the course),

- demonstrate understanding of the topic introduced as well as the necessary background material from our course,
- be written at a level that it could be understood by any of your classmates,
- be well-written, well-organized, and well-formatted,
- be 7-10 pages long and contain at least 3 pages of proof-writing (not consecutively, but the total amount of proof-writing should be at least 3 pages).[1] Points will be deducted if your paper is under 7 pages or if your paper is over 10 pages. The bibliography does not count towards your page count.

Your presentation can be done using slides or the chalk board. A good presentation will:

- introduce your topic to the class and give an overview of the big ideas,
- be understandable to your classmates (don't assume any knowledge outside of what we covered in class and remind us of any results/definitions as needed),
- be well-rehearsed and well-presented,
- (for slides talks) have easy-to-read slides that are not overcrowded with writing/pictures,
- (for chalk talks) have good board work that is organized with easy-to-read handwriting and with enough content written on the board,
- be 13-17 minutes long. If you end before 13 minutes or go over 17 minutes, you will lose points. Any individual presentation will be cut off at 20 minutes.[2]

Note: You **do not** have to give any detailed proofs in your presentation, and for many people, it will be best to only give proof outlines/summaries, or to skip proofs completely. Instead, you should give a summary of the big ideas in your project (e.g. What's your main theorem/computation? What tools do you need to introduce? Are there examples you can show us to help us understand the statement better?). You will not be able to present everything in your paper within 15 minutes, so don't try to do that! Your goal is to tell a coherent story in 15 minutes.

Partner Projects

You have the option to work alone or work with a partner on the final project. You'll state your preference in the "project proposal" form. I cannot guarantee you will get to work a partner (this will depend on everyone's responses on the proposal form). If you do work with a partner, you will write a paper together and present together. The requirements are modified for partner projects.

For the paper:

- Partners should write one paper that you submit together. It must be 12–15 pages long and contain at least 4 pages of proofs (not consecutively). Each partner is expected to significantly contribute to the research process, the writing process, and the editing processes.

[1] The requirements for partner papers are slightly different. See the next section.
[2] The requirements for partner presentations are slightly different. See the next section.

- Each partner will need to submit answers to a short Canvas assignment summarizing each person's contributions (this will be provided closer to the due date and be due by the end of finals week).

For the presentation:

- Partner presentations should be 25 minutes long (the acceptable range will be 23–27 minutes with a hard cut off at 30 minutes).
- Partners can either alternate slides, change speaker halfway, alternate sections, etc. It's up to you how you want to split up the presentation time, but each person must speak for at least 10 minutes.

Writing Expectations: Length, Format, Style

Length and Format

Projects should be completed using the "Project Template" on Canvas to ensure consistency between projects. A completed project should be between 7–10 (12–15 for partner projects) single-spaced pages that have been typed in LaTeX using 12-point font and 1-inch margins. The bibliography does not count towards the page count (so if your bibliography spills onto page 11, that's fine). You should correctly use theorem/definition environments, proof environments, sections, and figures. You are welcome to include hand-drawn figures as long as the drawing is clear and properly included into the tex document. Figures should be reasonably sized and readable, but also not take up an excessive amount of space in the paper.

Writing Style

Your project should include definitions, theorems, and exposition in between. You should weave together general discussion, definitions, lemmas/theorems/corollaries, and proofs. When you're writing your project, pretend you are writing a section in a textbook. The goal is to teach your reader about a new topic.
Here are some questions to ask yourself as you write: *How can you introduce the reader to a new definition? What examples (or non-examples) can you include to clarify parts of the definition? How can you motivate a theorem? What lemmas can you include to make a big proof more understandable?*

Citations

While preparing your project, you should consult and cite at least two resources, one of which can be our textbook. These resources can be textbooks, research papers, or expository papers. Examples of how to cite in LaTeX will be provided in a template on Canvas.

Other Formatting Considerations

A good project will be well-written and properly formatted. Sentences will start with capital letters and end with punctuation. All equations will be incorporated into full-sentences. Paragraph breaks will be used appropriately (you shouldn't have huge blocks of text without breaks, but you also shouldn't be putting every sentence of your proof on a new line). Both the tex code *and* the resulting pdf will have been proof-read many times to remove typos and formatting errors.

If you are unsure about formatting, you should look at the formatting of our textbook and the textbooks/papers you are consulting. You can also ask me for formatting advice as needed!

Academic Honesty: Getting External Help

You are allowed to consult external resources such as textbooks, papers, or websites/blogs. You are also allowed to talk with me or other students about the mathematics, writing, and any other topics in your project. But you must cite any resources you use in your paper, and you should also acknowledge any assistance you received from other students (there will be an "acknowledgements" section that you can use in the template).

You should not be using AI tools or LLMs for any reason while coming up with the mathematical ideas or writing style in your project. The only exception is that you <u>can</u> consult these tools or online forums for help with LaTeX commands and formatting.

Academic Honesty: Writing in Your Own Voice

You will be writing an expository paper. To learn about your topic, you will be reading textbooks, blogposts, papers, etc. You can take inspiration from articles you enjoy, but your final project must be written in your own voice. It should differ significantly in voice and structure from any of your consulted sources. For example, you might find a chapter in a book that covers your topic very well. This is great! You can consult this as you write, taking inspiration from what you think the author did well. But you should not just be rewriting line-by-line. Your project should have some different examples, different exposition, different ordering of topics, etc. If you are unsure about this, come talk to me!

Grade Breakdown

Your final project is worth 25% of your final grade, and that 25% is broken up as follows:

- **Project Preference Form (0.5%)**, due Sunday, April 13th at 11:59PM on Canvas.
- **Project Plan Form (0.5%)**, due Sunday, April 20th at 11:59PM on Canvas.
- **Presentation and Paper Outlines (1%)**, due Friday, May 2nd at 11:59PM on Canvas.
- **Practice Presentation Reflection (1%)**, due either Sunday, May 4th or Tuesday, May 6th at 11:59PM on Canvas (two days before your assigned presentation day).
- **Presentation (5%)**, either Tuesday, May 6th or Thursday, May 8th in class.

  - This will be broken into: 4% quality of your presentation, 1% your engagement during other students' presentations. The usual engagement policies apply.
- **Draft of Written Project (2%)**, due Friday, May 9th at 11:59PM on Canvas.
- **Final Submission of Project (15%)**, due Wednesday, May 14th at 11:59PM on Canvas.

More information about each of these is given below.

Project Preference and Project Plan Forms (1% of your final grade)

In the preference form you will rank the possible topics based on what you'd like to do. You will also indicate if you would prefer to work with a partner or to work alone.

Once you have your assigned topic, you will need to complete the "Project Plan Form" on Canvas. In this assignment, you will write a few sentences about what you are planning to cover in your project as well as give two possible sources. It's okay if these sources end up changing in your final written project, but I want to make sure you have sources in mind to get you started.

Presentation and Paper Outlines (1% of your final grade)

You will be required to submit an outline for your presentation and your paper by the end of Week 13 on Canvas. More information will be posted on Canvas.

Practice Presentation Reflection (1% of your final grade)

You will be required to complete a short "Practice Presentation Reflection" on Canvas that will be due two days before your presentation date. The goal of this is to ensure you practice your presentation in advance and reflect on what changes you need to make. Specific directions will be provided on Canvas.

Written Project Draft (2% of your final grade)

The draft is intended to check you are on track to complete your project by the deadline. In this draft, you need to submit at least **three pages of completed work** for individual projects (five pages for partner projects) that you will include in the final submission. These don't have to be the three first pages or even three consecutive pages–they could be the main theorem(s) and lemma(s) with their proofs, they could be the introduction and background sections, they could be some combination, etc. You just need to have three pages worth of complete work.

Presentation (5% of your final grade)

The presentation should be a well-rehearsed, well-written introduction to your topic. See "General Information" for more details. I will post a presentation rubric closer to the date, and your score on that rubric will be worth 4%. The other 1% will be based on your engagement during other students' presentations.

Final Written Submission (15% of your final grade)

The final submission should be a complete, well-written, and well-organized document. You should follow all guidelines given under "Writing Expectations" above. The final project will be scored out of 20 points, which will be broken down by quality of writing and proper formatting (5 points), breadth of material covered (5 points), and the precision/correctness of the mathematics presented (10 points).

Possible Project Topics

Below are a handful of project ideas for you to pick from.

1. The Fundamental Group of $\mathbb{S}^n$ for all $n \geq 1$

In class I stated that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ and gave a rough outline for how to prove this. In the first part of this project, you will provide a detailed proof of this theorem (this will likely require multiple lemmas). In the second part of this project, you will show $\mathbb{S}^n$ is simply connected for all $n \geq 2$. This will require you to consider so-called "space-filling" curves.

Possible resources:

- Our textbook, *Topology Through Inquiry* by Michael Starbird and Francis Su.
- *Algebraic Topology* by Allen Hatcher. Available for free on the author's website: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
- *A first course in Algebraic Topology* by Czes Kosniowski. Available for free download through the Grinnell library.
- *Topology* by James Munkres.

2. Covering Spaces

Algebraic topology can give us a way to answer difficult geometric questions using algebraic tools that are more accessible/computable. But there are also examples where we can use topological tools to answer algebraic questions. One beautiful example of this is through the theory of covering spaces and the subgroup correspondence theorem.

For a given space X, this theorem asserts that there is a correspondence between subgroups of $\pi_1(X)$ and "covers" of X, which are special spaces $\tilde{X}$ that come equipped with a surjective, continuous map $p : \tilde{X} \to X$. Thus we can learn things about the subgroup structure of $\pi_1(X)$ by studying possible covers of X, and vice versa.

In this project, you'll define covering spaces and prove some of the key lifting properties of covering spaces (see Theorems 13.4, 13.5, 13.7, 13.8, 13.9 from the text). You'll also define cover isomorphisms and state the subgroup correspondence.

From there you could do the following (note you can't do all of these options in one paper):

- Provide a proof of Theorem 13.28. This might take multiple lemmas.
- Define trees and graphs, and then use covering space theory to prove that every subgroup of a free group is free (see Corollary 13.14).
- Define 2-manifolds and gives some examples of 2-manifolds and covers of 2-manifolds by 2-manifolds. Prove (or just state) that a finite-sheeted cover of a compact 2-manifold is again a compact 2-manifold.

Possible resources:

- Our textbook, *Topology Through Inquiry* by Michael Starbird and Francis Su.
- *Algebraic Topology* by Allen Hatcher. Available for free on the author's website: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
- *A first course in Algebraic Topology* by Czes Kosniowski. Available for free download through the Grinnell library.
- *Topology* by James Munkres.

3. Category Theory

We've been studying the fundamental group and $\mathbb{Z}_2$-homology, which are examples of *algebraic invariants*. An algebraic invariant is something that assigns to each topological space an algebraic object (e.g. a group). Furthermore each continuous map between spaces gives a homomorphism between the corresponding groups. We can think of this as some type of "function" whose inputs are spaces and continuous maps, and whose outputs are groups and homomorphisms. But since our inputs/outputs are not coming from a set, we can't really call these "functions", but the analogy has still been a useful one to keep in mind.

This leads us to the following question: Is there another framework beyond sets and functions that we can use to more precisely and rigorously discuss algebraic invariants? The answer is yes! *Category theory* provides a framework to study and describe "functions between objects" where the "objects" aren't necessarily elements of a set. More formally, a category consists of a collection of objects together with a collection of morphisms (or maps) between pairs of objects. Here are a few familiar examples:

- The category **Grp** whose objects are groups and whose morphisms are group homomorphisms.
- The category **Ring** whose objects are rings and whose morphisms are ring homomorphisms.
- The category **Top** whose objects are topological spaces and whose morphisms are continuous maps. We can also consider the category $\mathbf{Top}_*$ whose objects are topological spaces with a chosen basepoint and whose morphisms are continuous maps that send basepoint to basepoint.

A *functor* is a map between categories, and so, for example, we can show π_1 is a functor from $\mathbf{Top}_*$ to **Grp**. In this project, you'll make all of these notions precise, reframing our algebraic invariants as functors between categories. Here are some ideas for this project:

- Define categories, functors, natural transformations. Discuss familiar examples (groups, rings, topological spaces, vector spaces, etc). You could also define "isomorphism of categories" and "equivalence of categories" and explore the difference between these notions.
- Define the category of chain complexes over $\mathbb{Z}_2$ and the category of simplicial complexes. Show that the functor that sends a simplicial complex to its $\mathbb{Z}_2$-simplicial chain complex is indeed a functor. Also discuss how "taking homology" gives a functor from chain complexes to $\mathbb{Z}_2$-vector spaces.
- The Van Kampen Theorem can be reframed categorically using the language "pushouts" (that is π_1 takes certain types of pushout diagrams in **Top** to pushout diagrams in **Grp**). Define "pushouts" and explain why our framing of Van Kampen's Theorem is equivalent to a statement about pushouts.
- A groupoid is a generalization of a group, and can be defined as a category in which every morphism is invertible. Instead of defining the fundamental group of a space, one can instead consider the "fundamental groupoid" of a space. Define groupoids (in categorical language) and the fundamental groupoid. Prove that the fundamental groupoid is equivalent to the fundamental group (as categories).

Possible resources:

- *Basic Category Theory* by Tom Leinster.
- *Category Theory in Context* by Emily Riehl. Available for free on the author's website: https://emilyriehl.github.io/books/. (Make sure you select the correct book! "Categorical Homotopy Theory" is a bit more advanced than needed for this project.)

4. Manifolds and the Invariance of Dimension

An *n-manifold* is a topological space in which every point has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$. These spaces "locally look like $\mathbb{R}^n$". Some familiar examples of 2-manifolds are the unit sphere in $\mathbb{R}^3$, the torus, and the Klein bottle. See Figure 1. In this project, you'll define

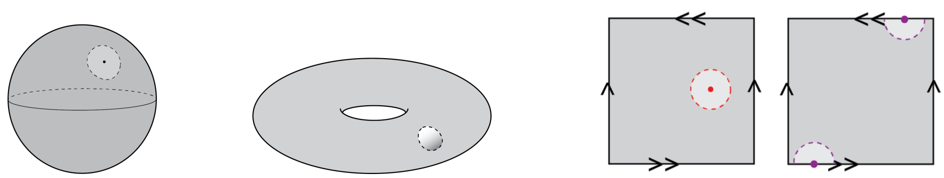

Figure 1: Examples of 2-manifolds.

manifolds, give some examples, and then prove the following *invariance of dimension* theorems using $\mathbb{Z}_2$-simplicial homology:

Theorem. The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$.

Theorem. A space cannot be both an n-manifold and an m-manifold if $n \neq m$.

Some other topics you could include in this project:

- Discuss and state (without proof) the classification of all compact 2-manifolds (this will require you to define the "connected sum" operation).
- Discuss and state (without proof) the embedding theorem, which states that any compact m-manifold can be embedded into $\mathbb{R}^N$ for some large N. (This will require you to define "embedding". You might give examples of embeddings of the Klein bottle or $\mathbb{R}P^2$ into $\mathbb{R}^4$.)
- We can generalize our definition of $\mathbb{R}P^2$ to define spaces $\mathbb{R}P^n$, which are quotient spaces of the unit sphere $\mathbb{S}^n$. Define $\mathbb{R}P^n$ (there are a few ways to do this) and show $\mathbb{R}P^n$ is an n-manifold. Show $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ for all $n \geq 2$.

Possible resources:

- Our textbook, *Topology Through Inquiry* by Michael Starbird and Francis Su.
- *Introduction to Topological Manifolds* by John M. Lee. Available for free download through the Grinnell library.
- *Topology* by James Munkres.

5. The Basics of $\mathbb{Z}$-Homology and the Hairy Ball Theorem

Suppose you had a rubber ball with rubber filaments sticking up (see photo of a "Koosh ball" below taken from Wikipedia). Is it possible to lay all of the fibers flat without creating a bald spot or

cowlick (also called a "hair whorl")? It turns out the answer is "no" and the proof of this fact uses algebraic topology!

The precise statement and theorem are about *nonzero vector fields on* $\mathbb{S}^n$. A vector field is a continuous map $V: \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that x and $V(x)$ are orthogonal for all $x \in \mathbb{S}^n$. We say a vector field is nonzero if $V(x) \neq 0$ for all $x \in \mathbb{S}^n$. For example, the map $V: \mathbb{S}^1 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (-y, x)$ gives a nonzero vector field on $\mathbb{S}^1$. See Figure 2 below. The "hairy ball theorem" says we can find a nonzero vector field if and only if n is odd:

Hairy Ball Theorem. There exists a nonzero vector field on $\mathbb{S}^n$ if and only if n is odd.

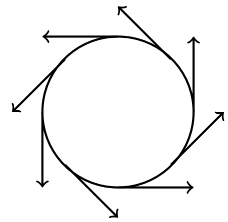

Figure 2: $V : \mathbb{S}^1 \to \mathbb{R}^2$, $(x, y) \mapsto (y, -x)$ (outputs are drawn as vectors starting at the input).

If we imagine the rubber Koosh ball as $\mathbb{S}^2$ and the filaments as vectors in $\mathbb{R}^3$, one for each point on the surface of the sphere, then we get a vector field by laying all of the hairs down flat. The "nonzero" part corresponds to not having any bald spots. The Hairy Ball Theorem tell us that, no, we cannot lay the hairs down flat without forming a bald spot!

In this project you will introduce the necessary algebraic topology tools to prove this theorem and then you will provide a proof. Here is one idea for how you might do this:

- Define $\mathbb{Z}$-simplicial homology and state the relevant theorems (similar to what we did or will do for $\mathbb{Z}_2$-simplicial homology in class).
- Compute $H_i(\mathbb{S}^n)$ and then define the degree of a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$. What is the degree of the antipodal map $x \mapsto -x$?
- Show that if there is a nonzero vector field on $\mathbb{S}^n$ then the identity map is homotopic to the antipodal map.
- Prove the Hairy Ball Theorem.

Possible resources:

- Our textbook, *Topology Through Inquiry* by Michael Starbird and Francis Su.
- *Algebraic Topology* by Allen Hatcher. Available for free on the author's website: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
- *A first course in Algebraic Topology* by Czes Kosniowski. Available for free download through the Grinnell library.

6. A Survey of Proofs about Simplicial $\mathbb{Z}_2$-Homology

In class I stated (or will state) a handful of theorems about simplicial $\mathbb{Z}_2$-homology without providing any proofs. In this project, you would prove these theorems. Here are some ideas for this project:

- Prove $\mathbb{Z}_2$ Mayer-Vietoris (Theorem 15.37). This will require you to prove some preliminary lemmas/theorems.
- Discuss barycentric subdivision (see Section 14.4) and prove the Simplicial Approximation Theorem (Theorem 14.33). Then explain how we use this fact to get induced maps for general (not necessarily simplicial) continuous maps between simplicial complexes. Then prove homotopy invariance.

Possible resources:

- Our textbook, *Topology Through Inquiry* by Michael Starbird and Francis Su.
- *Algebraic Topology* by Allen Hatcher. Available for free on the author's website: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
- *A first course in Algebraic Topology* by Czes Kosniowski. Available for free download through the Grinnell library.

7. Cell Complexes and Cellular Homology

Simplicial complexes are spaces built out of simplicies, and can be thought of as gluing together simplicies in a nice way. If we instead build spaces out of disks (again gluing things in a nice way) we get spaces called *cell complexes* or *CW complexes*. The different building blocks in dimensions 1, 2, and 3 are shown in Figure 3.

Figure 3: Simplicies and cells.

Once we have a notion of cell complexes, we can define cellular $\mathbb{Z}_2$-homology in a similar way as we did simplicial $\mathbb{Z}_2$-homology. One advantage of cellular homology is that we often need fewer cells compared to simplicies to build familiar spaces, and so the corresponding chain complex is easier to work with. In this project you will explore cellular homology.

Here are some ideas for this project:

- Define cellular homology for simplicial complexes following the method of our textbook. State (and maybe prove) that cellular and simplicial homology are isomorphic.
- Compute cellular homology for a bunch of spaces (see Exercise 15.49).
- Define cell complexes without referencing simplicial complexes. Do some examples and prove some properties of cell complexes (this is not in our textbook, so you'll need a different source for this part).

Possible resources:

- Our textbook, *Topology Through Inquiry* by Michael Starbird and Francis Su.
- *Algebraic Topology* by Allen Hatcher. Available for free on the author's website: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
- *A first course in Algebraic Topology* by Czes Kosniowski. Available for free download through the Grinnell library.

8. A Tool from Topological Data Analysis: Persistent Homology

Suppose we have a collection of data points $\mathbb{X}$ in $\mathbb{R}^d$. That is, suppose we have a finite subset $\mathbb{X} \subset \mathbb{R}^d$. See Figure 4 for an example in $\mathbb{R}^2$. If we look at Figure 4 and squint a bit, it seems like this collection of data has a "hole" in the center. The algebraic invariants we've studied during the second half of this course are good at detecting holes, so maybe we can use algebraic topology to detect this feature of our data!

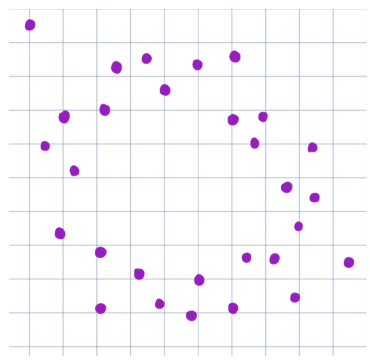

Figure 4: $\mathbb{X}$, some data points in $\mathbb{R}^2$.

Note that topologically the space $\mathbb{X}$ is boring: it is just a disjoint union of discrete points. Thus we don't learn anything by applying our invariants to $\mathbb{X}$ itself. Instead we associate a simplicial complex to the collection points and then take homology of this complex. One way to construct such a complex is by connecting data points with a 1-simplex whenever they are within some fixed distance $\epsilon > 0$ of one another. Then whenever the 1-simplices form a triangle, fill in that triangle with a 2-simplex, and so on. We get a family of complexes by varying ϵ, noting more simplicies will be added as ϵ grows. See Figure 5 for an example.

Persistent homology is interested in finding nonzero homology classes that "persist" for large intervals of ϵ. This gives us a way to study the "shape" of a data cloud, with the idea that classes that persist correspond to meaningful holes in the data. To learn a bit more, this 8-minute video gives an excellent introduction to persistent homology: `https://www.youtube.com/watch?v=h0bnG1Wavag&ab_channel=MatthewWright`

In this project, you will give an overview of the tools and techniques used in persistent homology. Some topics you could discuss/define/explore:

- What are the different simplicial complexes one can associate to a point cloud in $\mathbb{R}^d$ ("Cech complex", "Rips complex", "alpha complex")? How are these related to one another?
- What is a "filtration of complexes"? How do we use a filtration to get a "barcode"? How do we interpret this barcode?
- Can you do some examples with a small point cloud in $\mathbb{R}^2$ or $\mathbb{R}^3$? (See Page 43 of Magnus Bakke Botnan's notes, for example.)
- What are some examples of past/current/future applications of persistent homology?

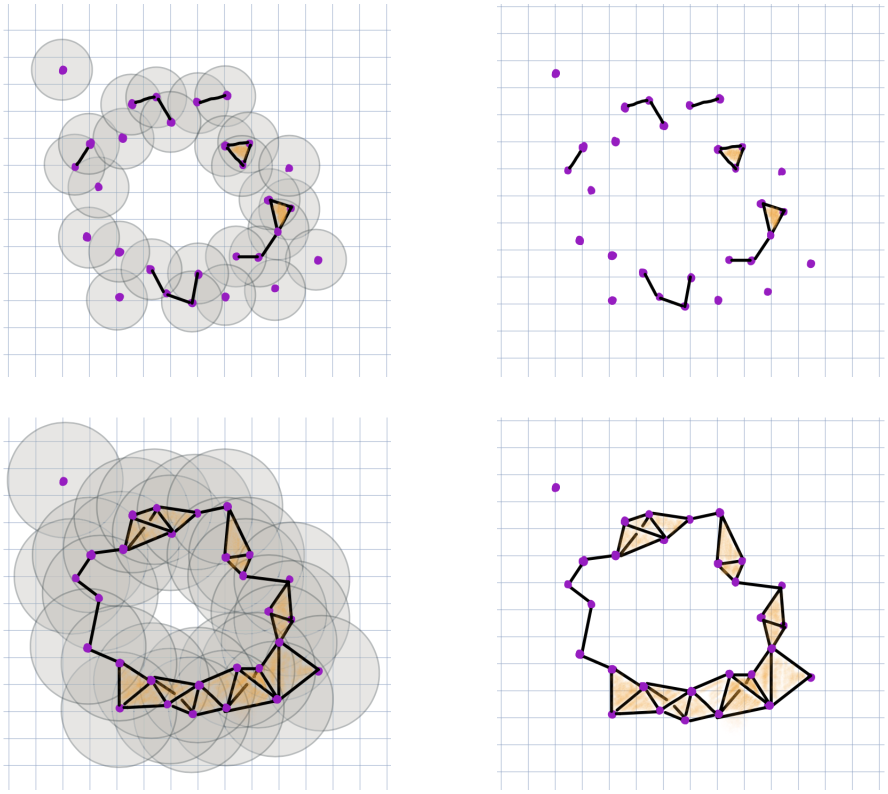

Figure 5: Connecting points for two different values of ϵ. The left grids show the ϵ-balls around each data point. The right grids show the resulting simplicial complexes.

Possible resources:

- Course lecture notes on "Topological Data Analysis" by Magnus Bakke Botnan. Available here on the author's website: https://www.few.vu.nl/~botnan/lecture_notes.pdf
- "Barcodes: The Persistent Topology of Data" by Robert Ghist. This is a short article from the Bulletin of the American Mathematical Society. Available on the AMS website here: https://www.ams.org/journals/bull/2008-45-01/S0273-0979-07-01191-3/S0273-0979-07-01191-3.pdf
- *Computational Topology: An Introduction* by Herbert Edelsbrunner and John Harer. Available online: https://www.maths.ed.ac.uk/~v1ranick/papers/edelcomp.pdf

Disclaimer: This could be a super fun and interesting project, especially for someone interested in applied mathematics or data analysis. But as a warning, there is not a clear path through this project as with the others. That is, there isn't one focal theorem or computation, and so you will need to put some work into deciding what story you'd like to tell and how you'll meet the project requirements. I am happy to meet with you and brainstorm ideas if you are interested in this project!