PARTIALLY ORDERED SETS, HASSE DIAGRAMS, AND LATTICES

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1. Partially ordered sets

Our main object of study this summer will be transfer systems, which are special relations on objects known as "lattices" which are examples of "partially ordered sets". Thus our story begins with the following definition.

Definition 1.1. A **partially ordered set** or **poset** is a set P with a relation $\preceq$ that satisfies the following properties:

- $\preceq$ is *reflexive*, i.e. for all $x \in P$, $x \preceq x$;
- $\preceq$ is *antisymmetric*, i.e. for all $x, y \in P$ if $x \preceq y$ and $y \preceq x$ then $x = y$; and
- $\preceq$ is *transitive*, i.e. for all $x, y, z \in P$ if $x \preceq y$ and $y \preceq z$ then $x \preceq z$.

We often write $(P, \preceq)$ for a poset with its relation or just write P when the relation is clear from context.

Here are a few examples of posets to get us started.

(1) Consider the natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ with the relation given by the usual less than or equal ordering $\leq$ so

$$0 \leq 1 \leq 2 \leq 3 \leq 4 \leq \dots$$

In a similar vein, if we take any natural number n and define $[n] := \{0, 1, 2, \dots, n\}$ then $[n]$ is a poset with ordering $0 \leq 1 \leq \dots \leq n$.[1]

(2) Let n be a positive integer and define D_n to be the set of positive divisors of n. Then D_n is a poset under the relation $d_1 \preceq d_2$ if and only if $d_1 \mid d_2$. For example $D_{12} = \{1, 2, 3, 4, 6, 12\}$ and we have $1 \preceq 2$, $2 \preceq 6$, $3 \preceq 6$. But note $2 \not\preceq 3$. We call this the **divisor poset** for n.

(3) For a positive integer n define B_n to be the set of all subsets of $\{1, 2, \dots, n\}$. Then B_n is a poset with the relation given by subset inclusion $\subseteq$. We call B_n a **Boolean lattice** (this naming convention will make more sense after Example 1.9).

Remark 1.2. We've been using the funny-looking less than or equals symbol $\preceq$ to emphasize the relation is not necessarily the usual "less than or equal" ordering coming from the real numbers. But it is typical for authors to still use $\leq$ for an abstract relation on a poset and to read the symbol as "less than or equal to". We will also adopt this convention from now on, writing $\leq$ instead of $\preceq$ for an abstract relation on a poset P. Be careful that $\leq$ does not always mean ordering of numbers though! (This is similar to how we use multiplicative notation $*$ or $\cdot$ and say "times" for an abstract binary operation in group theory.)

[1] Warning: Some authors write $[n]$ for the set $\{1, 2, \dots, n\}$. We are going to follow the convention in work of Balchin-Barnes-Roitzheim and have $[n] = \{0, 1, 2, \dots, n\}$.

Suppose $(P, \leq)$ is a poset. We adopt the following notation and terminology for convenience.

- For $x, y \in P$ we write $x \geq y$ if and only if $y \leq x$.
- For $x, y \in P$ we write $x < y$ if and only if $x \leq y$ and $x \neq y$.
- For elements $x, y \in P$ we say x and y are **comparable** if $x \leq y$ or $y \leq x$. Otherwise we say x and y are **incomparable**.

In the poset $(\mathbb{N}, \leq)$ any two elements are comparable, but this need not be true for general posets. For example $\{1,2\}$ and $\{2,3\}$ are elements of B_3 but $\{1,2\} \nsubseteq \{2,3\}$ and $\{2,3\} \nsubseteq \{1,2\}$. Thus $\{1,2\}$ and $\{2,3\}$ are incomparable elements of B_3.

Example 1.3. We can find more examples of posets from abstract algebra. Let G be a group and write $\mathrm{Sub}(G)$ for the set of all subgroups of G. Then $\mathrm{Sub}(G)$ has a relation given by subgroup inclusion (so $H \leq K$ if and only if H is a subset of K). This relation on $\mathrm{Sub}(G)$ forms a poset. Subgroup posets will be especially important for us when we investigate transfer systems.

As a concrete example, suppose $G = C_8 = \langle t \rangle$ where t is an element of order 8. (So C_8 is a cyclic group generated by t. Note C_8 has 8 elements.) Then

$$\mathrm{Sub}(C_8) = \{\{e\}, \{e, t^4\}, \{e, t^2, t^4, t^6\}, C_8\}.$$

Note any two elements from $\mathrm{Sub}(C_8)$ are comparable. What happens if $G = C_6 = \langle t \rangle$ where now t has order 6? Can you find all elements of $\mathrm{Sub}(C_6)$? Are any two elements comparable?

We introduce a few more helpful pieces of terminology and notation for posets.

Definition 1.4. Suppose $(P, \leq)$ is a poset. If Q is a subset of P then we say $(Q, \leq)$ is a **subposet** of $(P, \leq)$. *Note it is important that the relation on Q is the same as the relation on P.*

For any natural number n, note $[n]$ is a subposet of $\mathbb{N}$. Similarly for positive integers $n \leq k$, one can check that the Boolean lattice B_n is a subposet of B_k.

Consider the divisor poset D_{12}. Note the set D_{12} is a subset of $\mathbb{N}$, but it is not considered a subposet of $\mathbb{N}$ because the sets have different relations.

Definition 1.5. Let $(P, \leq)$ be a poset and $x, y \in P$. We write $[x, y]$ for the **interval from x to y** which is defined to be the set

$$[x, y] := \{z \in P \mid x \leq z \leq y\}.$$

Definition 1.6. Given posets $(P, \leq_P)$ and $(Q\ \leq_Q)$ we can form the **product poset** $P \times Q$ which consists of the set of all ordered pairs (x, y) where $x \in P$ and $y \in Q$ with relation $\leq$ given by

$$(x_1, y_1) \leq (x_2, y_2) \quad \Longleftrightarrow \quad x_1 \leq_P x_2 \text{ and } y_1 \leq_Q y_2.$$

(*It'd be good to pause here and think through why $P \times Q$ really is a poset!*)

Example 1.7. Consider $[1] \times [1]$. As set we have

$$[1] \times [1] = \{(0,0), (0,1), (1,0), (1,1)\}$$

and we have

$$(0,0) \leq (0,1) \leq (1,1) \quad \text{and} \quad (0,0) \leq (1,0) \leq (1,1),$$

while $(0,1)$ and $(1,0)$ are incomparable.

We can generalize the product construction to form the product poset of any finite number of posets. If we are taking the product of a poset P with itself n times, then we write P^n for the poset $\underbrace{P \times P \times \cdots \times P}_{n \text{ times}}$.

We end this section with a notion of what it means for two posets to be the "basically the same". As we often see in abstract algebra, "same" will mean there exists a bijection that preserves the poset structure.

Definition 1.8. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two posets. A function $f\colon P_1 \to P_2$ is an **isomorphism of posets** if f is a bijection and for all $x, y \in P_1$,

$$x \leq_1 y \iff f(x) \leq_2 f(y).$$

We say $(P_1, \leq_1)$ is **isomorphic** to $(P_2, \leq_2)$ if there exists an isomorphism of posets between them. In this case we write $P_1 \cong P_2$.

Example 1.9. We show $[1] \times [1] \cong B_2$. Recall B_2 is the set of all subsets of the two-element subset $\{1, 2\}$. We define a function $f\colon [1] \times [1] \to B_2$ by using each ordered pair $(x, y) \in [1] \times [1]$ as a "recipe" to build a subset of $\{1, 2\}$ in the following way. If $x = 0$ then we do not include 1. If $x = 1$ then we do include 1. Similarly if $y = 0$ then we do not include 2, and if $y = 1$ then we do include 2. Explicitly we have

$$f((0,0)) = \emptyset, \quad f((0,1)) = \{2\}, \quad f((1,0)) = \{1\}, \quad f((1,1)) = \{1,2\}.$$

Observe f respects the ordering of both posets, and so we have that f is a poset isomorphism. In the exercises below you'll generalize this to show $[1]^n \cong B_n$.

Example 1.10. Consider the poset $[3] = \{0, 1, 2, 3\}$ where $0 \leq 1 \leq 2 \leq 3$. Define a function

$$f\colon [3] \to \mathrm{Sub}(C_8), \qquad f(i) = \langle t^{2^{3-i}} \rangle.$$

That is, $0 \mapsto \{e\}$, $1 \mapsto \langle t^4 \rangle$, $2 \mapsto \langle t^2 \rangle$ and $3 \mapsto \langle t \rangle$. You can check that f does indeed give an isomorphism of posets.

Proposition 1.11. *The relation "is isomorphic to" is an equivalence relation on posets.*

Proof Outline. We need to prove that the relation "is isomorphic to" is reflexive, symmetric, and transitive. We outline how to show each step and leave the details to the reader.

To show reflexivity, it is enough to show the identity function is a poset isomorphism. To show symmetry, it is enough to show that if $f : (P_1, \leq_1) \to (P_2, \leq_2)$ is a poset isomorphism then the inverse function $f^{-1}\colon (P_2, \leq_2) \to (P_1, \leq_1)$ is also a poset isomorphism. Finally for transitivity, it is enough to show that if $f\colon (P_1, \leq_1) \to (P_2, \leq_2)$ and $g\colon (P_2, \leq_2) \to (P_3, \leq_3)$ are both poset isomorphisms then the composition $g \circ f\colon (P_1, \leq_1) \to (P_3, \leq_3)$ is a poset isomorphism. □

2. Hasse diagrams

Given a finite poset P there is a nice way to visualize the relations in P using a directed graph. One way to get such a graph is to have a vertex for each element in P and an edge from vertex x to vertex y whenever $x \leq y$. But if you try drawing such a graph for our examples, you'll see it can get messy pretty quickly. Furthermore a lot of the edges have redundant information. For example if $x \leq y$

and $y \leq z$ then we already know $x \leq z$ because $\leq$ is transitive, so the edge from x to z isn't providing any new information. We thus seek to produce a directed graph that includes a minimal amount of necessary information to encode P. This is done by recording what are known as "covering relations".

Definition 2.1. Suppose $(P, \leq)$ is a poset and $x, y \in P$ with $x \leq y$. We say $x \leq y$ is a **covering relation** if $x < y$ and there does not exist $z \in P$ with $x < z < y$. In this case we say y **covers** x or x is **covered by** y.

Given the covering relations on P we get all relations by noting that $x \leq y$ if and only if $x = y$, y covers x, or there exist elements $z_1, \ldots, z_n$ and a string of covering relations $x < z_1 < z_2 < \cdots < z_n < y$. Hence if you know all of the covering relations of a poset P, then you can determine all relations in P.

Definition 2.2. The **Hasse diagram** for a poset $(P, \leq)$ is the directed graph whose vertex set is P and whose edge set is given by the covering relations of P. Concretely, we have a directed edge $x \to y$ if and only if $x < y$ is a covering relation.

We will draw our Hasse diagrams so that bigger elements are on top or to the right of smaller elements, as you'll see in the examples below. *Our edges always flow up and to the right*, and thus we often don't include the arrows on the edges.

Example 2.3. The Hasse diagrams for [1], [3], and more generally, $[n]$ are shown in Figure 1. The Hasse diagrams for B_2 and B_3 are shown in Figure 2. We leave it as an exercise for you to fill in the labels on the vertices in these two diagrams.

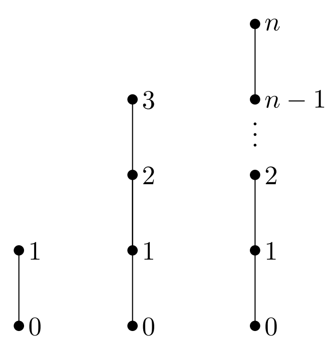


FIGURE 1. Hasse diagrams for [1], [3], $[n]$.

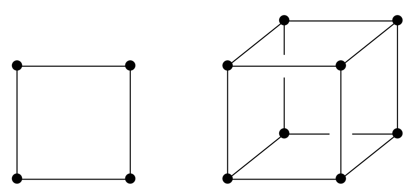


FIGURE 2. Hasse diagrams for B_2 and B_3

3. Lattices

We will be especially interested in lattices, which are posets that have nice notions of least upper bounds and greatest lower bounds. We make these ideas precise below.

Definition 3.1. Let $(P, \leq)$ be a poset and $x, y \in P$. A **join** of x and y is an element $z \in P$ such that $x \leq z$, $y \leq z$, and if $w \in P$ is any other element in P such that $x \leq w$ and $y \leq w$, then $z \leq w$.

Proposition 3.2. *For x, y in a poset P, if a join of x and y exists then it is unique.*

Proof. We leave this an exercise. □

Thus, if it exists, we refer to a join of x and y as *the* join of x and y, and we write $x \vee y$ for the join of x and y.

The join is also called the "least upper bound" of x and y (if you've taken analysis, you might recognize this phrase!). For example, in B_5 the join of $\{1, 2, 3\}$ and $\{2, 4\}$ is the set $\{1, 2, 3, 4\}$–this is the smallest set that contains both $\{1, 2, 3\}$ and $\{2, 4\}$. More generally the join of two elements in B_n is given by their union.

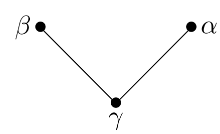
Definition 3.3. Let $(P, \leq)$ be a poset and $x, y \in P$. A **meet** of x and y is an element $z \in P$ such that $z \leq x$, $z \leq y$, and if $w \in P$ is any other element in P such that $w \leq x$ and $w \leq x$, then $w \leq z$.

Proposition 3.4. *For x, y in a poset P, if a meet of x and y exists then it is unique.*

Proof. We leave this an exercise. □

If it exists then we write $x \wedge y$ for the meet of x and y. The meet is also called the "greatest lower bound" of x and y. Going back to our example in B_5, we have that the meet of $\{1, 2, 3\}$ and $\{2, 4\}$ is the set $\{2\}$–this is the largest set that is a subset of both $\{1, 2, 3\}$ and $\{2, 4\}$. In general the meet of two elements in B_n is given by their intersection.

Note meets and joins need not always exist. Take for example the poset $P = \{\alpha, \beta, \gamma\}$ whose Hasse diagram is given below.



The meet of α and β is given by $\alpha \wedge \beta = \gamma$. But the join $\alpha \vee \beta$ does not exist because there are no elements comparable to both α and β.

Definition 3.5. A poset L is called a **lattice** if for all $x, y \in L$ the join $x \vee y$ and the meet $x \wedge y$ exist in L.

The poset $P = \{\alpha, \beta, \gamma\}$ given above is an example of a poset that is not a lattice.

Example 3.6. The following posets are examples of lattices.

(1) The Boolean lattice B_n is indeed a lattice. Meets are given by intersections and joins are given by unions.

(2) The poset $[n]$ is a lattice. The meet of two numbers is just given by the minimum while the join is given by the maximum.
(3) Given any positive integer n, the divisor poset D_n forms a lattice. How do you describe meets and joins in this case?
(4) Given a group G the subgroup poset $\mathrm{Sub}(G)$ forms a lattice. Recall the intersection of any two subgroups is again a subgroup. Thus meets are given by intersections. However it is not true that the union of any two subgroups is again a subgroup, so joins are a bit more complicated to describe. If H and K are subgroups of G such that at least one of them is normal, then the join is given by the set

$$HK = \{hk \mid h \in H, k \in K\}.$$

(One needs to prove this subset is actually a subgroup and is the smallest subgroup containing both H and K.) If both H and K fail to be normal, then the join is the subgroup generated by H and K, which can be described as the intersection of all subgroups of G that contain both H and K.

We end with one more definition and nice feature of lattices.

Definition 3.7. Given a poset $(P, \leq)$ we say an element $m \in P$ is a **minimal element** or **minimum** if for all $x \in P$, $m \leq x$. We say $m \in P$ is a **maximal element** or **maximum** if for all $x \in P$, $x \leq m$.

Proposition 3.8. *Let L be a finite lattice (so a lattice whose underlying set is finite). Then L has a unique maximal element and a unique minimal element.*

Proof. This is Exercise (12a). □

Given a finite lattice L we will often write $\top$ for the maximal element and $\perp$ for the minimal element.

4. Exercises

(1) Consider the Boolean lattice B_5. For each pair X, Y of sets below, list the sets in the interval $[X, Y]$.
   (a) $X = \{1, 2\}$, $Y = \{1, 2, 4, 5\}$
   (b) $X = \{1, 2\}$, $Y = \{2, 3, 4\}$
   (c) $X = \emptyset$, $Y = \{1, 2, 3\}$
(2) Fill in the vertex labels for the Hasse diagrams in Figure 2.
(3) Generalize Example 1.9 by showing for all positive integers n, $[1]^n$ is isomorphic to B_n.
(4) Let n be a positive integer. Draw the Hasse diagram for $[1] \times [n]$.
(5) Let K denote the Klein-4 group, so $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. Write down all elements of $\mathrm{Sub}(K)$. Then draw the Hasse diagram for $\mathrm{Sub}(K)$.
(6) (a) Draw the Hasse diagram for the divisor lattice D_{24}.
   (b) Let p and q be distinct prime numbers and let r be a positive integer. Draw the Hasse diagram for the divisor lattice $D_{p^r q}$.
(7) Let D_{24} denote the divisor lattice for 24. Prove that $[1] \times [3]$ is isomorphic to D_{24}.[2]

[2] If you have extra time, try to generalize this example in whatever way seems natural to you!

(8) Generalize Example 1.10 in the following way. Show for all primes p and for all positive integers n, $[n]$ is isomorphic to $\text{Sub}(C_{p^n})$. (You'll need to brush off some abstract algebra here. You might start by reviewing how to describe all subgroups of a cyclic group.)

(9) For a positive integer n describe how to find meets and joins in the divisor lattice D_n.

(10) Prove that if the posets $(P, \leq_P)$ and $(Q, \leq_Q)$ are lattices then $P \times Q$ is a lattice.

(11) Let n be an integer with $n \geq 2$. Write down a definition for the meet and then for the join of n elements from a poset P. (You should write a definition that doesn't reference the meet/join or two elements.)

(12) (a) Suppose P is a finite lattice. Prove that there exists a unique maximal and a unique minimal element in P. (This is Proposition 3.8.)

(b) Give an example of an infinite lattice that has a maximal element but no minimal element.